[13.7] SO(3) is the group of rotations of the unit sphere in 3-space. O(3) extends SO(3) by including reflections. **(A)** Show that SO(3) is a normal subgroup of O(3) and **(B)** show that it is the only proper normal subgroup.

**Note.** (B) is actually not true. There is one other proper normal subgroup of O(3). If 1 is the identity element (null rotation) of SO(3) and R is the reflection operator then **** ={ 1, R } is a normal subgroup of O(3). This is because if *g* is a (reflective or non-reflective) rotation, then *g*‑1 1 *g* = 1 and *g*‑1 R *g* = R (see Lemma 3). So we revise (B): show that O(3) has only two proper normal subgroups.

**Note:** In this proof we adopt the convention that *f g* represents rotating by *f* followed by *g*. So *f* R means to rotate and then reflect while R *f* means to reflect then rotate.

**Proof:** Penrose gives the hint: “What are the only sets in O(3) that are rotation invariant?”. The answer is simple. In Theorem 2 we show there are only 2 such sets: SO(3) and T. We begin with some preliminaries.

**Definitions:**

1. Let **S** be the unit sphere of ℝ3
2. Let **R** be the reflection operation on S
3. Let **T** = R[SO(3)] = {R*g*: *g* ∈ SO(3)} be the coset of reflective rotations in O(3)
   1. SO(3) and T are disjoint, and O(3) = SO(3) ∪ T
4. Let **1** be the identity of O(3), the null rotation

R is defined as an operation that reverses *xyz* orientation. It can be a reflection through the *xy-*plane, the *yz*-plane, or the *xz*-plane, or (preferred) a reflection through the origin, which is the composition of reflecting first through the *xy*‑plane, then the *yz*-plane, and then the x*z*-plane. Also, R-1 = R. If P is a point of S then PR = -P.

While SO(3) is a group, T is not. (It is only a coset.) For example, 1∉T. Also, if *t*1 and *t*2 belong to T, their composition *t*1 *t*2 ∉ T. Rather, *t*1 *t*2 ∈ SO(3). This is because R is applied twice in the expression *t*1 *t*2. In fact, any expression with an even number of reflections belongs to SO(3), and it belongs to T if the number of reflections is odd.

If *t*∈ T, there are elements *s*1, *s*2 ∈ SO(3) such that *t* = R *s*1 and *t* = *s*2 R. The former is true by definition of T. The latter is seen to be true by setting *s*2 = R *s*1 R.

We need the following theorem to answer Penrose’s invariance question.

**Theorem 1.**

1. Let *s*1, *s*2 ∈ SO(3). Then ∃ *s*3 ∈ SO(3) such that *s*2 = *s*3 *s*1.
2. Let *t*1, *t*2 ∈ T. Then ∃ *s* ∈ SO(3) such that *t*2 = *s* *t*1.

**Proof:** (a) *s*3 = *s*2 *s*1-1. (b) *s* = *t*2 *t*1-1. *s* ∈ SO(3) because this expression has 2 reflections, an even number. 

**Theorem 2** (Answer to Penrose’s question): SO(3) and T are the only proper subsets of O(3) that are rotation invariant.

**Proof:** SO(3) is rotation-invariant because applying a rotation to any rotation in SO(3) yields another rotation, an element of SO(3). SO(3) has no proper subset A that is rotation-invariant because, by Theorem 1a, given any *s*1∈ A and *s*3∉ A, one can find a rotation *s*2 such that *s*1 *s*2 = *s*3; i.e., *s1* is rotated out of A.

Similarly, T is rotation invariant because applying a rotation to any element of T remains in T because only 1 reflection has occurred. Like SO(3), T cannot have a proper subset A that is rotation invariant because, by Theorem 1b, from any *t*1 ∈ A one can obtain any *t*2∉ A by applying a rotation. 

**Part A**

**Theorem A:** SO(3) is a normal subgroup of O(3)

**Proof**. First, SO(3) is clearly a group because it contains the identity; inverses of are just reverse rotations (which are still rotations); and the product (composition) of 2 rotations is a rotation (closed). To see that it is normal, let *s* ∈ SO(3). If *g*∈ SO(3), then *g* -1 *s* *g* ∈ SO(3) because it is the composition of 3 rotations. If *g*∈ T, then *g*‑1 *s* *g*∈ SO(3) because the expression involves 2 reflections. So *g*‑1 SO(3) *g* = SO(3). Left multiplying both side by *g* yields SO(3) *g* = *g* SO(3). So SO(3) is normal by Penrose’s definition of normal.

The above proof doesn’t use Penrose’s hint, so here is a proof that does. Since SO(3) is rotation invariant, *g* SO(3) ⊆ SO(3). By Theorem 1, *g* SO(3) ⊇ SO(3). Therefore *g* SO(3) = SO(3). Similarly, because SO(3) is rotation invariant, SO(3) *g* = SO(3). Thus *g* SO(3) = SO(3) *g* which proves SO(3) is normal.  

**Part B**

We need the following lemma a few times so it is worth introducing here.

**Lemma 1:** If *g* is a 90° rotation and *h* is a non-zero rotation having an axis of rotation perpendicular to the axis of rotation of *g*, then *f* = *g* -1 *h* *g* is a rotation having an axis of rotation perpendicular to both *g* and *h*.

**Proof:** WLOG let

* *g* be a 90° counter-clockwise rotation about the *z*‑axis and
* *h* be a rotation of angle ** about the *x*-axis.

To show *f* is a rotation about the *y*-axis, it suffices to show that all points on the *y*‑axis are fixed during rotation *f*.

Consider the point (0,*y*,0) on the *y*-axis. Since *g-1*spins the *xy*-plane 90° clockwise, (0,*y*,0) *g*‑1 = (*y*,0,0). Since points on the *x*-axis are fixed during rotation *h*, (*y*,0,0) *h* = (*y*,0,0) . Finally, since *g* spins the *xy*-plane 90° counter-clockwise, (*y*,0,0) *g*‑1 =(0,*y*,0). That is (0,*y*,0) is a fixed point of the rotation *f*. 

**Aside:** I did not see an explanation in Road To Reality of why O(3) is non-Abelian. However, using the descriptions of *g*-1 and *h* given in Lemma 1, it is easy to show that *g*-1*h* ≠ *h g*-1:

(0,1,0) *g*-1*h* = (1,0,0) *h* = (1,0,0)

(0,1,0) *h* *g*-1= (0, cos **, sin **) *g*-1 = (cos **, 0, sin **) 

**Lemma 2:** Let *g*, *h* ∈SO(3) and let *h* have rotation angle *θ*. Then *f*  = *g* ‑1 *h* *g* has the same angle of rotation *θ*  as *h* (although a possibly different axis of rotation).

*f*

O

*x*

*y*

*z*

Q’

M’’

**

P’

*f*

Q

M

**

P

O

*x*

*y*

*z*

*h*

*g*

**Proof:** We use the property that rotations in ℝ3 preserve rigid bodies. WLOG assume *g* is a counter-clockwise rotation about the *z*-axis. Let P∈ S. Let  represent the axis of revolution of *h* and let *h* rotate point P to a point M. PQM represents the rotation angle *θ* of *h*.



Set P’ = P *f*, Q’= Q *f*, and M’= M *f*.

Define the rigid body B to be the union of  with the angle PQM. (It looks like line segment with 2 spikes.) Rotation *f* moves B as a rigid body so that it becomes the union of with angle P’Q’M’. The angle remains *θ*.



We are now in a position to find the angle and axis of rotation of *f*. Start with a point P’ of S. We know that P’ = P *f*, that *h* rotates P an amount *θ* about OQ to point M, that M’= M *f* and Q’= Q *f*. So we know that *f* rotates point P’ to M’ by an angle *θ* about axis OQ’.

(See my version 2 solution to [13.7] for a more rigorous, equation-based proof of this lemma.) 

It was shown in problem [12.17] that SO(3) is group isomorphic to the (solid) 3‑ball **** of radius *π* in which antipodal points on the surface of **** are identified. Points of **** can be represented as *θ* (*a*, *b*, *c*) = (*θ* *a*, *θ* *b*, *θ* *c*) where *θ* is the angle of rotation and (*a*, *b*, *c*) is a unit vector in the direction of the axis of rotation.

**Definition:** For 0 ≤ *θ* ≤ *π* let **S**** be the sphere of radius *θ* in **.** S** consists of angle *θ* rotations about each axis of rotation.

**Theorem 3:** In SO(3), if a rotation *h* with rotation angle *θ* belongs to a normal subgroup H, then S** ⊆ H.

**Proof:** Let *g* ∈SO(3). Let F = { *f* =*g*‑1 *h g* : *g* ∈ SO(3) } ⊆ H. From Lemma 2, F ⊆ S**. Thus if *f* ∈ F, it has rotation angle *θ* and can be expressed as *f*= *θ* (*a, b, c*) for some (*a, b, c*) where *a*2 + *b*2 + *c*2 = 1. To prove F = S**, we must show for every point (*a*, *b*, *c*) of the unit sphere S in ℝ3 that *θ* (*a, b, c*) ∈ F. We do this by building up subgroup F starting from the single element *h*.

WLOG we can let *h* = *θ* (0, 0, 1). Consider a great circle arc from *g0* =  (0, 0, 1) to *g1* =  (0, 1, 0 ) on the surface of ball . Since *g*0 has same axis of rotation as *h*, then *f*0 = *g*0‑1 *h g*0 = *h* = *θ*  (0, 0, 1). Since *g*1 has an axis of rotation perpendicular to that of *h*, by Lemma 1 *f*1 = *g*1‑1 *h g*1 has an axis of rotation perpendicular to both *h* and *g*1. That is, *f*1 = *θ* (1, 0, 0). Thus as

*x*

*y*

*z*

**0****(0,0,1)

****(1,0,0)

*f* arc

*S*

*x*

*y*

*z*

*g*0 **½****(0,0,1)

*g***½****(0,1,0)

*g* arc

*g* = (0, sin **, cos **) moves along the arc in **** from *g*0 to *g*1 (i.e., from ** = 0 to ** = ), *f* = *g* ‑1 *h g* = (sin **, 0, cos **) moves along the great circle arc in S*θ* from *f*0 to *f*1.

Now sweep the *g* arc in a clockwise 360° circle as indicated in the figure. This sweeps out the northern hemisphere on the surface of the ball . The corresponding *f* arc sweeps out the northern hemisphere of S**. That is, for every point (*a*, *b*, *c*) on the northern hemisphere of the unit sphere S there is a *θ* rotation *f* on the northern hemisphere of S*θ* such that *f* = *θ* (*a, b, c*) = *g*‑1 *h g* ∈ F.

For the southern hemisphere, note that *h*‑1 = *θ* (0, 0, -1) ∈ H since H is a group. The *g* and *f* arcs based on *h*‑1 similarly sweep out the southern hemispheres.

Thus for every point (*a*, *b*, *c*) on the unit sphere, *θ* (*a, b, c*) equals either *g*‑1 *h* *g* or *g*‑1 *h*‑1*g* for some *g* ∈ SO(3), proving that *θ* (*a, b, c*) ∈ F and concluding the proof. (This proof is performed more rigorously using equations in my version 2 proof.)



**Theorem 4:** SO(3) has no proper normal subgroup.

**Proof:** Let H be a non-trivial normal subgroup of SO(3). ∃ 1 ≠ *h* ∈ H having some rotation angle *θ*. By Theorem 3, S** ⊆ H.

We take products of elements in S** to grow H beyond S**. Let *g*, *h* ∈ S** and *f* = *h g*. The maximum possible angle for *f* is 2** and the minimum is 0. The maximum is obtained when *g* = *h* and the minimum is obtained when *g* = *h*- 1. By letting *g* take a path in from *h* to *h*- 1 we generate a path ofpoints *f* = *h g*in H having every possible angle ** from 0 to 2**. By Theorem 3, S** ⊆ H for 0 ≤ **≤ 2**.Thus every point of **** in the closed ball of radius 2** belongs to H. If 2*θ ≥ ,* then we are done. If not, starting from sphere S2** we similarly grow H to include the closed ball of radius 4**, then 8**, … Eventually we obtain that all of SO(3) = ****⊆ H. 

**Lemma 3:** Let *g* ∈ SO(3). Then *g*-1R*g* = R.

**Proof:** Let P be a point on the unit sphere S. Let Q = P*g*-1. Then ‑Q = QR = P*g*‑1R and P*g*‑1R*g* = (‑Q)*g* = ‑(Q*g*) = ‑P = PR. 

**Theorem B:** SO(3) and **** ={ 1,R } are the only proper normal subgroups of O(3).

**Proof:** Let H be a nontrivial normal subgroup of O(3) such that H ≠ SO(3) and H ≠ **** . We need to show that H = O(3).

Claim: There is an element *t* ∈ H ∩ T such that *t* ≠ R:

By Theorem 4, ∃ *t0* ∈ H ∩ T. If *t*0 ≠ R, the claim is true. If *t*0 = R, since H ≠ ****, H contains another element. If that element is in T, the claim is true. If the other element is *s*0 ∈ SO(3), we can assume *s*0 ≠ 1 since H ≠ ****. Set *t* = *s*0R. Then *t* ∈ H ∩ Tand *t* ≠ R.

*t* 2 ∈ SO(3). Suppose for the moment that *t* 2 ≠ 1. Then *t*2 ∈ SO(3) ⋂ H ⇒ SO(3) ⊆ H by Theorem 4. Also, ∃ *s*∈ SO(3) such that *t* = *s* R. Since *t* ≠ R then *s* ≠ 1.

Claim: T ⊆ H:

Let *t*1 ∈ T. ∃*s*1 ∈ SO(3) such that *t*1 = *s*1 R. Let *s*2 = *s*1 *s* ‑1∈ SO(3). Then *s*1 = *s*2 *s*. Since *s*2 ∈ SO(3) ⊂ H, *t*1 = *s*1 R = *s*2 *s* R = *s*2 *t* ∈ H. Thus T ⊆ H.

Since SO(3) ⊆ H, we have O(3) = SO(3)∪T ⊆ H. Therefore H = O(3).

Unfortunately if *s* has a 180° rotation angle, then *s*2 = 1 and thus *t* 2 = 1, and the above argument doesn’t quite hold. (Note: *t* 2 = 1 because if P is a point, then P*t*2 = P*s*R*s*R =  [P*s*]R*s*R = [‑P*s*]*s*R = -Ps2R = ‑PR = P). However, everything in the above argument remains true except that we haven’t proved SO(3) ⊆  H. Once we prove this, we are done.

Let *g* be a 90° rotation about an axis perpendicular to the axis of *s* and let *s*3 = *g*‑1*sg*. By Lemma 1, the rotation axis of *s*3 is perpendicular to that of *s*. Let *t*3 = *g*‑1 *t* *g* ∈ H. We have

*t*3 =*g*‑1*s*R *g* = *g*‑1 *s* (*g* *g*-1)R *g* = (*g*‑1 *s g*) (*g*-1 R *g*) = *s*3 R

by Lemma 3. Hence the axis of rotation of *t* is perpendicular to that of *t*3. (See footnote[[1]](#footnote-1).) Let *s*4 = *t*t*3.* Because inverses have the same axis of rotation, *t*3 ≠ *t* -1 and so *s*4 ≠ 1. Because H is a group, *s*4 ∈ H. Thus, by Theorem 4, SO(3) ⊆ H, completing the proof. 

1. The axis of rotation of the reflective rotations *t* and *t*3 can be considered to be located in the reflected unit sphere. They point in the opposite directions from the axes of rotation of *s* and is*3*, respectively. Thus, if the axes of *s* and *s*3 are perpendicular, then so are the axes of *t* and *t*3. [↑](#footnote-ref-1)